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A functional equation originating from quadratic forms ☆

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Abstract

In this paper, we obtain the general solution and the stability of the 2-variable quadratic functional equation

$$f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2f(y,w).$$

The quadratic form $f(x,y)=ax^2+bxy+cy^2$ is a solution of the above functional equation.

Keywords: Solution; Stability; 2-Variable quadratic functional equation

1. Introduction

A mapping f is called a *quadratic form* if there exist $a,b,c\in\mathbb{R}$ such that

$$f(x,y)=ax^2+bxy+cy^2$$

for all $x,y\in X$.

In this paper, let X and Y be real vector spaces. For a mapping $f:X\times X\to Y$, consider the 2-variable quadratic functional equation:

$$f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2f(y,w). \tag{1}$$

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When $X = Y = \mathbb{R}$, the quadratic form $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) := ax^2 + bxy + cy^2$ is a solution of (1).

For a mapping $g : X \to Y$, consider the quadratic functional equation:

$$g(x + y) + g(x - y) = 2g(x) + 2g(y). \tag{2}$$

In 1989, J. Aczél [1] solved the solution of Eq. (2). Later, many different quadratic functional equations were solved by numerous authors [2–5].

In this paper, we investigate the relation between (1) and (2). And we find out the general solution and the generalized Hyers–Ulam stability of (1).

2. Results

The 2-variable quadratic functional equation (1) induces the quadratic functional equation (2) as follows.

Theorem 1. *Let $f : X \times X \to Y$ be a mapping satisfying* (1) *and let $g : X \to Y$ be the mapping given by*

$$g(x) := f(x, x) \tag{3}$$

for all $x \in X$, then g satisfies (2).

Proof. By (1) and (3),

$$\begin{aligned} g(x + y) + g(x - y) &= f(x + y, x + y) + f(x - y, x - y) \\ &= 2f(x, x) + 2f(y, y) \\ &= 2g(x) + 2g(y) \end{aligned}$$

for all $x, y \in X$. □

Example 1. Let X be a real algebra and $D : X \to X$ a derivation on X. Define a mapping $f : X \times X \to X$ by

$$f(x, y) := D(xy) = xD(y) + D(x)y$$

for all $x, y \in X$. Then f satisfies (1). Define a mapping $g : X \to X$ by

$$g(x) := D\bigl(x^2\bigr) = xD(x) + D(x)x$$

for all $x \in X$. Then g satisfies (3). By Theorem 1, g satisfies (2).

The quadratic functional equation (2) induces the 2-variable quadratic functional equation (1) with an additional condition.

Theorem 2. *Let $a, b, c \in \mathbb{R}$ and $g : X \to Y$ be a mapping satisfying* (2). *If $f : X \times X \to Y$ is the mapping given by*

$$f(x, y) := ag(x) + \frac{b}{4}\bigl[g(x + y) - g(x - y)\bigr] + cg(y) \tag{4}$$

for all $x, y \in X$, then f satisfies (1). *Furthermore,* (3) *holds if $a + b + c = 1$.*

Proof. By (2) and (4),

$$\begin{aligned}
&f(x+y,z+w)+f(x-y,z-w)\\
&\quad=ag(x+y)+\frac{b}{4}\big[g(x+y+z+w)-g(x+y-z-w)\big]+cg(z+w)\\
&\qquad+ag(x-y)+\frac{b}{4}\big[g(x-y+z-w)-g(x-y-z+w)\big]+cg(z-w)\\
&\quad=2ag(x)+2ag(y)+\frac{b}{2}\big[g(x+z)+g(y+w)\big]\\
&\qquad-\frac{b}{2}\big[g(x-z)+g(y-w)\big]+2cg(z)+2cg(w)\\
&\quad=2ag(x)+\frac{b}{2}\big[g(x+z)-g(x-z)\big]+2cg(z)\\
&\qquad+2ag(y)+\frac{b}{2}\big[g(y+w)-g(y-w)\big]+2cg(w)\\
&\quad=2f(x,z)+2f(y,w)
\end{aligned}$$

for all $x, y \in X$.

Letting $x = y = 0$ and $y = x$ in (2), respectively,

$$g(0)=0 \quad \text{and} \quad g(2x)=4g(x)$$

for all $x \in X$. By (4) and the above two equalities,

$$\begin{aligned}
f(x,x)&=ag(x)+\frac{b}{4}\big[g(2x)-g(0)\big]+cg(x)\\
&=(a+b+c)g(x)\\
&=g(x)
\end{aligned}$$

for all $x \in X$. □

Example 2. Consider the function $g:\mathbb{R}^2 \to \mathbb{R}$ given by $g(\mathbf{x}) := \mathbf{x}^T A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$, where A is a 2×2 real matrix. Then we can easily show that g satisfies (2). Let $a, b, c \in \mathbb{R}$ and define

$$f(\mathbf{x},\mathbf{y}) := ag(\mathbf{x})+\frac{b}{4}\big[g(\mathbf{x}+\mathbf{y})-g(\mathbf{x}-\mathbf{y})\big]+cg(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Then one can obtain the function $f:\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(\mathbf{x},\mathbf{y}) = \begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}^T \begin{pmatrix} a & \frac{b}{2}\\ \frac{b}{2} & c\end{pmatrix}\begin{pmatrix}A\mathbf{x}\\ A\mathbf{y}\end{pmatrix}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. By Theorem 2, the function f satisfies (1).

Example 3. Let M_n be the algebra of $n \times n$ real matrices. Consider the mapping $g: M_n \to M_n$ given by $g(A) := A^2$ for all $A \in M_n$. Then we can easily show that g satisfies (2). Let $a, b, c \in \mathbb{R}$ and define

$$f(A,B) := aA^2 + bA \circ B + cB^2,$$

where $A \circ B$ the Jordan product $\frac{1}{2}(AB + BA)$ of A and B for all $A, B \in M_n$. Then the mapping $f: M_n \times M_n \to M_n$ satisfies (4). By Theorem 2, the mapping f satisfies (1).

In the following theorem, we find out the general solution of the 2-variable quadratic functional equation (1).

Theorem 3. *A mapping* $f : X \times X \to Y$ *satisfies* (1) *if and only if there exist two symmetric bi-additive mappings* $S_1, S_2 : X \times X \to Y$ *and a bi-additive mapping* $B : X \times X \to Y$ *such that*

$$f(x, y) = S_1(x, x) + B(x, y) + S_2(y, y)$$

for all $x, y \in X$.

Proof. We first assume that f is a solution of (1). Define $f_1, f_2 : X \to Y$ by $f_1(x) := f(x, 0)$ and $f_2(x) := f(0, x)$ for all $x \in X$. One can easily verify that f_1, f_2 are quadratic. By [1], there exist two symmetric bi-additive mappings $S_1, S_2 : X \times X \to Y$ such that $f_1(x) = S_1(x, x)$ and $f_2(x) = S_2(x, x)$ for all $x \in X$. Define $B : X \times X \to Y$ by

$$B(x, y) := f(x, y) - \bigl[f(x, 0) + f(0, y)\bigr] \tag{5}$$

for all $x, y \in X$. Then B is bi-additive. Indeed, by (1) and (5), we obtain that

$$\begin{aligned}
&B(x_1 + x_2, y)\\
&\quad = f(x_1 + x_2, y) - \bigl[f(x_1 + x_2, 0) + f(0, y)\bigr]\\
&\quad = f(x_1 + x_2, y) - \frac{1}{2}\bigl[f(x_1 + x_2, y) + f(x_1 + x_2, -y)\bigr]\\
&\quad = \frac{1}{2}\bigl[f(x_1 + x_2, y) - f(x_1 + x_2, -y)\bigr]\\
&\quad = f(x_1 + x_2, y) - \frac{1}{2}\bigl[f(x_1 + x_2, y) + f(x_1 + x_2, -y)\bigr]\\
&\quad = f(x_1 + x_2, y) - \bigl[f(x_1 + x_2, 0) + f(0, y)\bigr]\\
&\quad = \frac{1}{2}\bigl[2f(x_1 + x_2, y) + 2f(0, y) - 2f(x_1 + x_2, 0)\bigr] - 2f(0, y)\\
&\quad = \frac{1}{2}\bigl[f(x_1 + x_2, 2y) - f(x_1 + x_2, 0)\bigr] - 2f(0, y)\\
&\quad = \frac{1}{2}\bigl[f(x_1 + x_2, 2y) + f(x_1 - x_2, 0)\bigr] - \frac{1}{2}\bigl[f(x_1 + x_2, 0) + f(x_1 - x_2, 0)\bigr] - 2f(0, y)\\
&\quad = f(x_1, y) + f(x_2, y) - f(x_1, 0) - f(x_2, 0) - 2f(0, y)\\
&\quad = f(x_1, y) - \bigl[f(x_1, 0) + f(0, y)\bigr] + f(x_2, y) - \bigl[f(x_2, 0) + f(0, y)\bigr]\\
&\quad = B(x_1, y) + B(x_2, y)
\end{aligned}$$

for all $x_1, x_2, y \in X$. Similarly $B(x, y_1 + y_2) = B(x, y_1) + B(x, y_2)$ for all $x, y_1, y_2 \in X$.

Conversely, we assume that there exist two symmetric bi-additive mappings $S_1, S_2 : X \times X \to Y$ and a bi-additive mapping $B : X \times X \to Y$ such that $f(x, y) = S_1(x, x) + B(x, y) + S_2(y, y)$ for all $x, y \in X$. Since B is bi-additive and S_1, S_2 are symmetric bi-additive,

$$\begin{aligned}
&f(x + y, z + w) + f(x - y, z - w)\\
&\quad = S_1(x + y, x + y) + B(x + y, z + w) + S_2(z + w, z + w)\\
&\qquad + S_1(x - y, x - y) + B(x - y, z - w) + S_2(z - w, z - w)\\
&\quad = S_1(x, x) + 2S_1(x, y) + S_1(y, y) + B(x, z) + B(x, w) + B(y, z) + B(y, w)
\end{aligned}$$

$$
\begin{aligned}
&+ S_2(z,z) + 2S_2(z,w) + S_2(w,w)\\
&+ S_1(x,x) - 2S_1(x,y) + S_1(y,y) + B(x,z) - B(x,w) - B(y,z) + B(y,w)\\
&+ S_2(z,z) - 2S_2(z,w) + S_2(w,w)\\
&= 2\bigl[S_1(x,x) + B(x,z) + S_2(z,z)\bigr] + 2\bigl[S_1(y,y) + B(y,w) + S_2(w,w)\bigr]\\
&= 2f(x,z) + 2f(y,w)
\end{aligned}
$$

for all $x, y, z, w \in X$. $\square$

Example 4. Consider the function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given in Example 2. By Theorem 3, there exist two symmetric bi-additive mappings $S_1, S_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and a bi-additive mapping $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(\mathbf{x}, \mathbf{y}) = S_1(\mathbf{x}, \mathbf{x}) + B(\mathbf{x}, \mathbf{y}) + S_2(\mathbf{y}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. In fact

$$S_1(\mathbf{x}, \mathbf{y}) = \frac{a}{2}\bigl(\mathbf{x}^T A\mathbf{y} + \mathbf{y}^T A\mathbf{x}\bigr), \qquad S_2(\mathbf{x}, \mathbf{y}) = \frac{c}{2}\bigl(\mathbf{x}^T A\mathbf{y} + \mathbf{y}^T A\mathbf{x}\bigr)$$

and

$$B(\mathbf{x}, \mathbf{y}) = \frac{b}{2}\bigl(\mathbf{x}^T A\mathbf{y} + \mathbf{y}^T A\mathbf{x}\bigr)$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

Let Y be complete and let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying

$$\tilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \varphi\bigl(2^j x, 2^j y, 2^j z, 2^j w\bigr) < \infty \tag{6}$$

for all $x, y, z, w \in X$.

Theorem 4. *Let $f : X \times X \to Y$ be a mapping such that*

$$\bigl\| f(x+y, z+w) + f(x-y, z-w) - 2f(x,z) - 2f(y,w) \bigr\| \leqslant \varphi(x, y, z, w) \tag{7}$$

for all $x, y, z, w \in X$. Then there exists a unique 2-variable quadratic mapping $F : X \times X \to Y$ such that

$$\bigl\| f(x,y) - F(x,y) \bigr\| \leqslant \tilde{\varphi}(x, x, y, y) \tag{8}$$

for all $x, y \in X$. The mapping F is given by

$$F(x,y) := \lim_{j\to\infty} \frac{1}{4^j} f\bigl(2^j x, 2^j y\bigr)$$

for all $x, y \in X$.

Proof. Letting $y = x$ and $w = z$ in (7), we have

$$\left\| f(x,z) + \frac{1}{4}\bigl[f(0,0) - f(2x, 2z)\bigr] \right\| \leqslant \frac{1}{4}\varphi(x, x, z, z)$$

for all $x, z \in X$. Thus we obtain

$$\left\| \frac{1}{4^j} f\left(2^j x, 2^j z\right) - \frac{1}{4^{j+1}}\left[f(0,0) + f\left(2^{j+1}x, 2^{j+1}z\right)\right] \right\| \leqslant \frac{1}{4^{j+1}} \varphi\left(2^j x, 2^j x, 2^j z, 2^j z\right)$$

for all $x, z \in X$ and all j. Replacing z by y in the above inequality, we see that

$$\left\| \frac{1}{4^j} f\left(2^j x, 2^j y\right) - \frac{1}{4^{j+1}}\left[f(0,0) + f\left(2^{j+1}x, 2^{j+1}y\right)\right] \right\| \leqslant \frac{1}{4^{j+1}} \varphi\left(2^j x, 2^j x, 2^j y, 2^j y\right)$$

for all $x, y \in X$ and all j. For given integers l, m $(0 \leqslant l < m)$, we get

$$\left\| \frac{1}{4^l} f\left(2^l x, 2^l y\right) - \frac{1}{4^m}\left[f(0,0) + f\left(2^m x, 2^m y\right)\right] \right\| \leqslant \sum_{j=l}^{m-1} \frac{1}{4^{j+1}} \varphi\left(2^j x, 2^j x, 2^j y, 2^j y\right) \tag{9}$$

for all $x, y \in X$. By (9), the sequence $\{\frac{1}{4^j} f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{4^j} f(2^j x, 2^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x, y) := \lim_{j\to\infty} \frac{1}{4^j} f\left(2^j x, 2^j y\right)$$

for all $x, y \in X$. By (7), we have

$$\begin{aligned} &\left\| \frac{1}{4^j} f\left(2^j(x+y), 2^j(z+w)\right) + \frac{1}{4^j} f\left(2^j(x-y), 2^j(z-w)\right) \right. \\ &\quad \left. - \frac{2}{4^j} f\left(2^j x, 2^j z\right) - \frac{2}{4^j} f\left(2^j y, 2^j w\right) \right\| \leqslant \frac{1}{4^j} \varphi\left(2^j x, 2^j y, 2^j z, 2^j w\right) \end{aligned}$$

for all $x, y, z, w \in X$ and all j. Letting $j \to \infty$ and using (6), we see that F satisfies (1). Setting $l = 0$ and taking $m \to \infty$ in (9), one can obtain the inequality (8). If $G : X \times X \to Y$ is another 2-variable quadratic mapping satisfying (8), we obtain

$$\begin{aligned} &\left\| F(x, y) - G(x, y) \right\| \\ &\quad = \frac{1}{4^n} \left\| F\left(2^n x, 2^n y\right) - G\left(2^n x, 2^n y\right) \right\| \\ &\quad \leqslant \frac{1}{4^n} \left\| F\left(2^n x, 2^n y\right) - f\left(2^n x, 2^n y\right) \right\| + \frac{1}{4^n} \left\| f\left(2^n x, 2^n y\right) - G\left(2^n x, 2^n y\right) \right\| \\ &\quad \leqslant \frac{2}{4^n} \tilde{\varphi}\left(2^n x, 2^n x, 2^n y, 2^n y\right) \\ &\quad \to 0 \quad \text{as } n \to \infty \end{aligned}$$

for all $x, y \in X$. Hence the mapping F is the unique 2-variable quadratic mapping, as desired. □

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